Symmetry is an incredibly powerful tool to help simplify calculations.
But, symmetry also plays a more fundamental role in determining the type of dynamics in certain physical theories. For example special relativity is nothing more than a statement about the symmetries of spacetime. And the SM forces can be seen to arise as the consequence of certain symmetries. We will discuss both of these in due time.

Your first exposure to symmetry was probably <u>static</u> type, e.g. $\triangle \xRightarrow{\circlearrowleft 120^\circ} \triangle$ (geometry, shapes, etc.)

Static symmetries are easy to visualize, but ...
We will be more interested in <u>dynamical</u> symmetries, e.g. Lagrangian: $L \xRightarrow{\text{some transformation}} L' = L$

No matter what type of symmetry we consider, the spirit is the same, i.e. we enact a <u>transformation</u> on something and afterwards that something looks the same.

Now naively it may seem that our something must only be built out of things which themselves are invariant. If this were the case it would be terribly restrictive. Fortunately, we can build an invariant something out of pieces which are not invariant so long as we combine them in an appropriate way, e.g. we can build a rotationally invariant scalar from vector components with a dot product!

So our preliminary focus will be on describing transformations. We will come back to making sure they are symmetries of a Lagrangian a bit later.

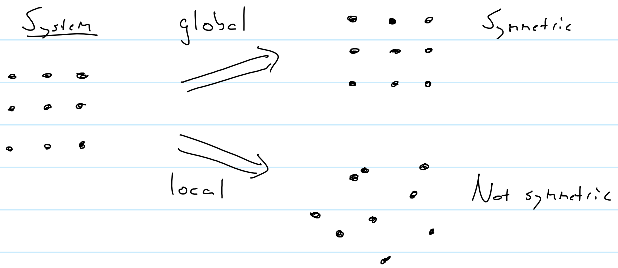
Transformations come in many different types: global, local, discrete, continuous, finite, infinite, compact, non-compact, internal, spacetime

To clarify most of these words we will look at static symmetry examples:

Global vs. Local

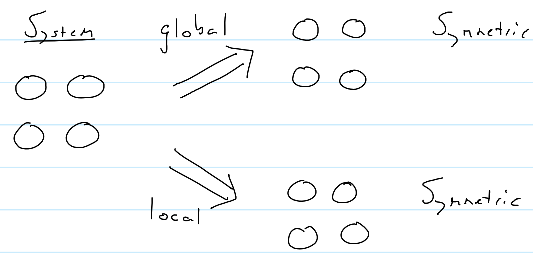
Example 1:

Transformation = translate each dot



Example 2:

Transformation = rotate each circle in plane



Note: If a system is symmetric under local transformations then it is automatically symmetric under global transformations, but the reverse is not true!

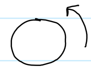
Discrete vs. Continuous

Discrete (finite or infinite)

Example 1:  $R_{120°}$ finite since $\{1, R_{120}, R_{240}\}$

Example 2: $-\infty \leftarrow \cdot \cdot \cdot \cdot \cdot \cdot \cdot \rightarrow +\infty$ (T_1, T_2) infinite since $\{T_1, T_2, T_3, \ldots\}$

Continuous (compact or noncompact)

Example 1:  R_θ $\theta \in [0, 2\pi)$ compact

Example 2: $-\infty \leftarrow \overline{\quad\quad\quad} \rightarrow +\infty$ (T_d) $d \in (-\infty, \infty)$ noncompact

Spacetime vs. Internal

If we coordinatize spacetime, then spacetime transformations also change coordinates while internal transformations do nothing to the coordinates.

Note: Special Relativity is associated with spacetime symmetries.
The strong, weak and electromagnetic forces are associated with internal symmetries.

For our purposes we can treat transformations mathematically using the concepts of groups and representations.

A group G is a collection of elements $\{A, B, \cdots\}$ with a composition $\bullet$ that satisfies:

1. Closure - if $A, B \in G \Rightarrow A \bullet B \in G$
2. Identity - there is some $I \in G$ such that $I \bullet A = A$ for any $A \in G$
3. Inverse - for any $A \in G$ there is an $A^{-1} \in G$ such that $A^{-1} \bullet A = I$
4. Associativity - $A \bullet (B \bullet C) = (A \bullet B) \bullet C$

These will be very important in building invariants!

We could add commutativity, i.e. $A \bullet B = B \bullet A$, in which case we have an abelian group, but we actually need groups that don't commute, i.e. they are non-abelian.

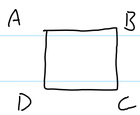
If we can take a subset of the elements of a group and they form a group themselves then this is a subgroup of the original group. Note: subgroups always have to include the identity and inverses and have to be careful to remain closed!

We can abstractly specify a group, e.g. Rotations in 2D with composition that we add the rotation angles.

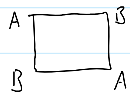
But more often we think (and calculate) in terms of how the group transformations act on things. These are called <u>representations</u> of the group.

A single group can often have many different representations. Some are more useful since the fully illustrate the content of the group, these are called <u>faithful representations</u>.

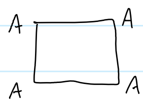
Example: G = Rotations in plane by 90° w/ usual composition (addition of angles).

Rep r_1: 

$\{ I, R_{90}, R_{180}, R_{270} \}$ This is the only faithful representation!

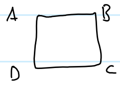
Rep r_2: 

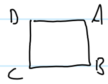
$\{ I, R_{90} \}$ This is a degenerate representation.

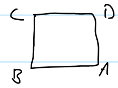
Rep r_3: 

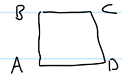
$\{ I \}$ This is the highly degenerate Identity representation.

We can often work with representations where the transformations act linearly using matrices:

If: 

$=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ 

$=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ 

$=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ 

$=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

then $I=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $R_{90}=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$, $R_{180}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$, $R_{270}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$

Notice that the matrices behave as expected, e.g. $R_{90}R_{180}=R_{270}$, etc.

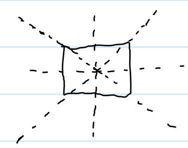
Note: This is an abelian group since any $RR'=R'R$.
In this case we utilized column matrices with real components. Sometimes we will make use of complex components.

For finite groups we can construct a "multiplication" table.

Example: For our earlier group G

	I	R_{90}	R_{180}	R_{270}
I	I	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	I
R_{180}	R_{180}	R_{270}	I	R_{90}
R_{270}	R_{270}	I	R_{90}	R_{180}

Note:
- For an abelian group the multiplication table must be symmetric across the diagonal $\diagdown$.
- In this example $\{I, R_{180}\}$ would be a subgroup, but $\{I, R_{90}\}$ would not!
- If we thought of the square in 3D, we could extend the group with four more transformations. These would be $180°$ rotations around the axes:



If two discrete groups have the same multiplication table then they are <u>isomorphic</u>.

	I	R_π
I	I	R_π
R_π	R_π	I

2 Rot. in 2D

	E	O
E	E	O
O	O	E

E, O w/ $+$

	1	-1
1	1	-1
-1	-1	1

$1, -1$ w/ $\times$

$$\mathbb{Z}_2 : \{I, g\} \text{ w/ } g^2 = I$$

For continuous groups we can't form a multiplication table so we have to work harder to compare them.

So far we have a formal way to describe how elements of a representation transform under an element of a group. But how can we build an invariant?

Given some thought, it might seem that combining two objects which transform "oppositely" would give an invariant. In fact this is exactly what we do!

We will take a cue from the familiar dot product, i.e. $\vec{v} \cdot \vec{w} = \#$ or $(v_1\ v_2\ v_3)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$

For any matrix representation r we can form the <u>dual representation</u> $\tilde{r}$ as follows:

If $A \in G$ then $r \to Ar$, $\tilde{r} \to (A^{-1})^T \tilde{r}$.

Then if we form $\tilde{r}^T r \to (A^{-1^T} \tilde{r})^T A r = \tilde{r}^T A^{-1^{T^T}} A r = \tilde{r}^T A^{-1} A r = \tilde{r}^T r$

In our example: $r = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\tilde{r} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

If we choose $A = \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in r$ and $\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \in \tilde{r}$ then:

$$\tilde{r}^T r = (e\ f\ g\ h)\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longrightarrow \left[\underbrace{\begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}}_{A^{-1}}{}^T \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right]^T \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= ea + fb + gc + hd$$

$$= \left[\begin{pmatrix} h \\ e \\ f \\ g \end{pmatrix} \right]^T \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix} = (h\ e\ f\ g)\begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix} = hd + ea + fb + gc$$

same !!

Note: We can do this for complex representations as well, but since Lagrangians must be real, we only want real invariants so we form $\tilde{r}^\dagger r$ instead.

$\dagger = *^T$